Fourth-order finite difference scheme with boundary value methods for space fractional diffusion equation

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Outline

Introduction

- In the last decade or so, the catholicity of anomalous diffusion phenomena in the real world has led to the fractional diffusion equation (FDE). FDEs arise in research topics including continuous time random walk (Wang et al. 11'), porous media transmission (Benson et al. 01'), entropy (Povstenko et al. 15'), hydrology (Benson et al. 00'), Brownian motion (Benson et al. 00'), physics (Sokolov et al. 02') and image processing (Bai et al. 07').

Introduction

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- Boundary value methods (BVMs) are constructed as the unconditionally stable and are high-accuracy schemes for solving ordinary difference equations (ODEs). Unlike Runge-Kutta or other initial value methods (IVMs), BVMs achieve the advantage of both good stability and high-order accuracy.

Outline

Riemann-Liouville fractional derivatives

Definition

Suppose $u(x)$ is an integrable function defined on (a, b), (a, b is finite or ∞), $\alpha > 0$, $(1 < \alpha < 2)$. Define

$$ {}_aD_x^\alpha u(x) = \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_a^x \frac{u(\eta)d\eta}{(x-\eta)^{\alpha-1}}, \tag{1} $$

$$ {}_xD_b^\alpha u(x) = \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_x^b \frac{u(\eta)d\eta}{(\eta-x)^{\alpha-1}}, \tag{2} $$

are left and right Riemann-Liouville fractional derivatives with α order, respectively.

Space fractional diffusion equation

Improve our former numerical method (Gu et al. 15') by utilizing a new compact spatial semi-discretization of the following FDE, and its convergence $\mathcal{O}(\Delta t^4 + h^4)$.

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = K_{1\,a}D_x^{\alpha} u(x,t) + K_{2\,x}D_b^{\alpha} u(x,t) + f(x,t), & \\ u(x,0) = u_0(x), & x \in [a,b], \\ u(a,t) = \phi_a(t), \quad u(b,t) = \phi_b(t), & t \in [0,T], \end{cases} \tag{3}$$

The diffusion coefficients K_1 and K_2 are nonnegative constants and they satisfy $K_1^2 + K_2^2 \neq 0$. If $K_1 = 0$, then $\phi_a(t) \neq 0$ and $K_2 = 0$, then $\phi_b(t) \neq 0$. Moreover, u_0, ϕ_a and ϕ_b are given sufficiently smooth functions. In the following analysis of the numerical method, we suppose that (3) has an unique and sufficiently smooth solution.

Outline

Earlier work

For the discretizaiton of Riemann-Liouville fractional derivative, we can briefly introduce,

- shifted Grünwald formula (Meerschaet and Tadjeran 04', Meerschaet and Tadjeran, 06')
- weighted and shifted Grünwald formula (Sun et al. 15', Deng et al. 15')
- midpoint approximation formula (Li et al. 16')

Hint: In above items, we almost consider the numerical approximations, who are unconditionally stable.

Fourth-order spatial discretization of space FDE

According to the above description, we establish a compact spatial semi-discretization scheme with the help of fourth-order weighted and shifted Grünwald difference (WSGD) operator, who is unconditionally stable. We have the following spatial discretized form of (3) ($j = 1, 2, \cdots, n_x$):

$$P_x u_j' = K_{1L}D_h^{\alpha} u_j + K_{2R}D_h^{\alpha} u_j + P_x f_j, \qquad j = 1, 2, \ldots, n_x, \qquad (4)$$

Fourth-order spatial discretization of space FDE

where $h=\frac{b-a}{n_x+1}$, $u_j'=\frac{du_j}{dt}$, $P_x=1+h^2b_2^\alpha\delta_x^2$, $\delta_x^2u_i=\frac{u_{i-1}-2u_i+u_{i+1}}{h^2}$,

$$_LD_h^\alpha u_j=\frac{1}{h^\alpha}\sum_{k=0}^{j+1}\omega_k^{(\alpha)}u_{j-k+1},\quad {_RD_h^\alpha} u_j=\frac{1}{h^\alpha}\sum_{k=0}^{n_x-j+2}\omega_k^{(\alpha)}u_{j+k-1},$$

the coefficients $\omega_0^{(\alpha)}=\mu_1g_0^{(\alpha)},\omega_1^{(\alpha)}=\mu_0g_0^{(\alpha)}+\mu_1g_1^{(\alpha)}$ and $\omega_k^{(\alpha)}=\mu_1g_k^{(\alpha)}+\mu_0g_{k-1}^{(\alpha)}+\mu_{-1}g_{k-2}^{(\alpha)}$, $k=2,\ldots,n_x+1$,

$$\mu_1=(1+\alpha)(2+\alpha)/12,\quad \mu_0=-(-2+\alpha)(2+\alpha)/6,$$
$$\mu_{-1}=(-2+\alpha)(-1+\alpha)/12,\quad b_2^\alpha=(4+\alpha-\alpha^2)/24.$$

Fourth-order spatial discretization of space FDE

Rewrite Eq.(4) as the following matrix product form:

$$[P_\alpha \mathbf{u}(t) + \mathbf{d}(t)]' = B_\alpha \mathbf{u}(t) + \mathbf{e}_1(t) + \mathbf{q}(t),$$

where $\mathbf{u}(t) = [u_1, u_2, \ldots, u_{n_x}]^T$, $B_\alpha = \frac{1}{h^\alpha}(K_1 W_\alpha + K_2 W_\alpha^{\mathrm{T}})$, W_α and P_α are, respectively, given

$$W_\alpha = \begin{pmatrix} \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & 0 & \cdots & 0 & 0 \\ \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & 0 & \cdots & 0 \\ \vdots & \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \omega_{n_x-1}^{(\alpha)} & \ddots & \ddots & \ddots & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} \\ \omega_{n_x}^{(\alpha)} & \omega_{n_x-1}^{(\alpha)} & \cdots & \cdots & \omega_2^{(\alpha)} & \omega_1^{(\alpha)} \end{pmatrix}$$

Fourth-order spatial discretization of space FDE

$$P_\alpha = \begin{pmatrix} 1-2b_2^\alpha & b_2^\alpha & & & \\ b_2^\alpha & 1-2b_2^\alpha & b_2^\alpha & & \\ & & \cdots & & \\ & & b_2^\alpha & 1-2b_2^\alpha & b_2^\alpha \\ & & & b_2^\alpha & 1-2b_2^\alpha \end{pmatrix},$$

$$\mathbf{d}(t) = [b_2^\alpha \phi_a(t), 0, \ldots, 0, b_2^\alpha \phi_b(t)]^{\mathrm{T}}, \ \mathbf{q}(t) := P_\alpha \mathbf{f}(t) + \mathbf{e}_2,$$

$$\mathbf{e}_1(t) := \frac{1}{h^\alpha} \begin{pmatrix} [K_1\omega_2^{(\alpha)} + K_2\omega_0^{(\alpha)}]\phi_a(t) + K_2\omega_{n_x+1}^{(\alpha)}\phi_b(t) \\ 0 \\ \vdots \\ 0 \\ [K_1\omega_0^{(\alpha)} + K_2\omega_2^{(\alpha)}]\phi_b(t) + K_1\omega_{n_x+1}^{(\alpha)}\phi_a(t) \end{pmatrix}$$

Fourth-order spatial discretization of space FDE

$\mathbf{f}(t) = [f_1, f_2, \dots, f_{n_x}]^T$ and $\mathbf{e}_2 = [b_2^\alpha f_0, 0, \dots, 0, b_2^\alpha f_{n_x+1}]^T$. To sum up, the quasi-compact spatial semi-discretization of FDEs (3) can be written as following system of ODEs:

$$\begin{cases} P_\alpha \frac{d\mathbf{u}(t)}{dt} = B_\alpha \mathbf{u}(t) + \mathbf{g}(t), \\ \mathbf{u}(t_0) = \mathbf{u}_0, \qquad t \in [t_0, T], \end{cases} \tag{5}$$

where

$$\mathbf{u}_0 = [u_0(x_1), u_0(x_2), \dots, u_0(x_{n_x})]^{\mathrm{T}}$$

$$\mathbf{g}(t) = \mathbf{e}_1(t) - \mathbf{d}'(t) + \mathbf{q}(t).$$

Outline

The BVMs

In comparison with the other initial value solvers, BVMs additional to having high-order accuracy, are unconditionally stable methods. A fourth-order BVM approximation of (5) can be obtained by $k = 3$ and $\gamma = 2$. Thus, (5) can be write as

$$(A \otimes P_\alpha - \Delta t B \otimes B_\alpha)\mathbf{u} = \Delta t (B \otimes I_{n_x})\mathbf{g} + \Delta t(\mathbf{b}_0 \otimes (B_\alpha \mathbf{u}_0 + \mathbf{g}_0)) - \mathbf{a}_0 \otimes P_\alpha \mathbf{u}_0,$$

where I_{n_x} is a $n_x \times n_x$ identity matrix and

$$\mathbf{u} = [u_1(t_1), u_2(t_1), \ldots, u_{n_x}(t_1), u_1(t_2), u_2(t_2), \ldots, u_{n_x}(t_2), \ldots, u_1(t_m), u_2(t_m), \ldots, u_{n_x}(t_m)]^T,$$

$$\mathbf{g} = [g_1(t_1), g_2(t_1), \ldots, g_p(t_1), g_1(t_2), g_2(t_2), \ldots, g_{n_x}(t_2), \ldots, g_1(t_m), g_2(t_m), \ldots, g_{n_x}(t_m)]^T,$$

$$\mathbf{g}_0 = [g_1(t_0), g_2(t_0), \ldots, g_{n_x}(t_0)]^T$$

The BVMs

$$A = \begin{bmatrix} 9/24 & 9/24 & -1/24 & & & \\ -9/12 & 9/12 & 1/12 & & & \\ -1/12 & -9/12 & 9/12 & 1/12 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & -1/12 & -9/12 & 9/12 & 1/12 \\ & & 1/24 & -9/24 & -9/24 & 17/24 \end{bmatrix}$$

$$B = \begin{bmatrix} 3/4 & 0 & & \\ 1/2 & 1/2 & & \\ & \ddots & \ddots & \\ & & 1/2 & 1/2 \\ & & 3/4 & 1/4 \end{bmatrix}$$

$\mathbf{a}_0 = [-17/21, -1/12, 0, \cdots, 0]^T$, and $\mathbf{b}_0 = [1/4, 0, \cdots, 0]^T$.

Numerical results

Example 1. Consider Eq.(3) with two specified positive coefficients K_1, K_2, $(a, b) \times (0, T] = (0, 1) \times (0, 1]$ and the source term

$$f(x,t) = -10^3 e^{-t}\Big[x^5(1-x)^5 + \frac{\Gamma(6)}{\Gamma(6-\alpha)}\Big(K_1 x^{5-\alpha} + K_2(1-x)^{5-\alpha}\Big) - \frac{5\Gamma(7)}{\Gamma(7-\alpha)}\Big(K_1 x^{6-\alpha} + K_2(1-x)^{6-\alpha}\Big) + \frac{10\Gamma(8)}{\Gamma(8-\alpha)}\Big(K_1 x^{7-\alpha} + K_2(1-x)^{7-\alpha}\Big) - \frac{10\Gamma(9)}{\Gamma(9-\alpha)}\Big(K_1 x^{8-\alpha} + K_2(1-x)^{8-\alpha}\Big) + \frac{5\Gamma(10)}{\Gamma(10-\alpha)}\Big(K_1 x^{9-\alpha} + K_2(1-x)^{9-\alpha}\Big) - \frac{\Gamma(11)}{\Gamma(11-\alpha)}\Big(K_1 x^{10-\alpha} + K_2(1-x)^{10-\alpha}\Big)\Big].$$

Then the exact solution is $u(x,t) = 10^3 e^{-t} x^5 (1-x)^5$, and we also have $u(x,0) = 10^3 x^5(1-x)^5$ and $u(0,t) = u(1,t) = 0$.

Numerical results

Table: The average (error) obtained for Example 1 with $h = \Delta t$, $K_1 = 0.2$ and $K_2 = 0.8$.

	$\alpha = 1.2$		$\alpha = 1.5$		$\alpha = 1.8$		$\alpha = 1.9$	
$n_x + 1$	Error	C-order	Error	C-order	Error	C-order	Error	C-order
8	5.8254e-03	–	4.1872e-03	–	2.5910e-03	–	1.8399e-03	–
16	2.3352e-04	4.6407	2.3384e-04	4.1624	1.8605e-04	3.7998	1.4671e-04	3.6486
32	1.1798e-05	4.3070	1.3656e-05	4.0979	1.1224e-05	4.0510	8.8649e-06	4.0487
64	6.6915e-07	4.1400	7.9940e-07	4.0945	6.6564e-07	4.0757	5.2678e-07	4.0728
128	3.9902e-08	4.0678	4.8269e-08	4.0497	4.0636e-08	4.0339	3.2894e-08	4.0013

Table: The average (error) obtained for Problem 1 with $h = \Delta t$, $K_1 = 0.6$ and $K_2 = 0.4$.

	$\alpha = 1.2$		$\alpha = 1.4$		$\alpha = 1.6$		$\alpha = 1.8$	
$n_x + 1$	Error	C-order	Error	C-order	Error	C-order	Error	C-order
8	2.9697e-03	–	2.7394e-03	–	2.7920e-03	–	2.2070e-03	–
16	1.7352e-04	4.0971	1.7828e-04	3.9416	1.8872e-04	3.8870	1.7235e-04	3.6786
32	1.0851e-05	3.9992	1.1780e-05	3.9197	1.2196e-05	3.9517	1.0612e-05	4.0216
64	6.8416e-07	3.9873	7.3818e-07	3.9962	7.5726e-07	4.0094	6.4865e-07	4.0321
128	4.4417e-08	3.9452	4.7288e-08	3.9644	4.7887e-08	3.9831	4.0729e-08	3.9933

Numerical results

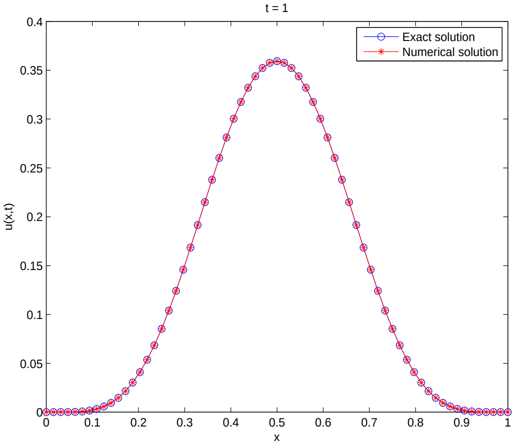

Figure: The behavior solution from the proposed methods at $t = 1$ with $h = \Delta t = 1/64$, $\alpha = 1.2$, $K_1 = 0.2$ and $K_2 = 0.8$.

Numerical results

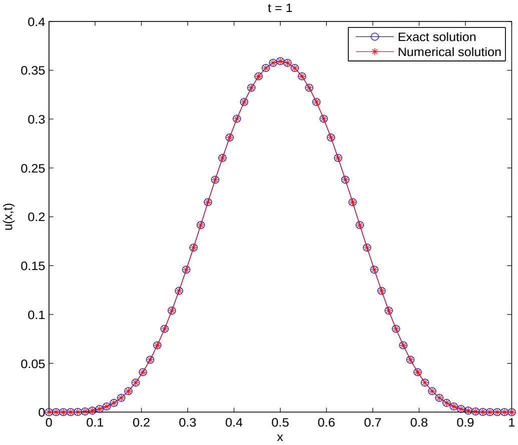

Figure: The behavior solution from the proposed methods at $t = 1$ with $h = \Delta t = 1/64$, $\alpha = 1.8, K_1 = 0.6$ and $K_2 = 0.4$.

Numerical results

Example 2. Consider Eq.(3) with two specified positive coefficients K_1, K_2, $(a,b)\times(0,T]=(0,1)\times(0,1]$ and the source term

$$
\begin{aligned}
f(x,t) = & \cos(t+1)x^4(2-x)^4 - \sin(t+1)\Big[\frac{\Gamma(9)}{\Gamma(9-\alpha)}\Big(K_1 x^{8-\alpha} \\
& + K_2(2-x)^{8-\alpha}\Big) - \frac{8\Gamma(8)}{\Gamma(8-\alpha)}\Big(K_1 x^{7-\alpha} + K_2(2-x)^{7-\alpha}\Big) \\
& + \frac{24\Gamma(7)}{\Gamma(7-\alpha)}\Big(K_1 x^{6-\alpha} + K_2(2-x)^{6-\alpha}\Big) - \frac{32\Gamma(6)}{\Gamma(6-\alpha)}\Big(K_1 x^{5-\alpha} \\
& + K_2(2-x)^{5-\alpha}\Big) + \frac{16\Gamma(5)}{\Gamma(5-\alpha)}\Big(K_1 x^{4-\alpha} + K_2(2-x)^{4-\alpha}\Big)\Big].
\end{aligned}
$$

In addition, the exact solution is $u(x,t)=\sin(t+1)x^4(2-x)^4$. Also, we have $u(x,0)=\sin(1)x^4(2-x)^4$ and $u(0,t)=u(2,t)=0$.

Numerical results

Table: The average (error) obtained for Problem 2 with $h = \Delta t$, $K_1 = 0.3$ and $K_2 = 0.7$.

$n_x + 1$	$\alpha = 1.2$		$\alpha = 1.5$		$\alpha = 1.8$		$\alpha = 1.9$	
	Error	C-order	Error	C-order	Error	C-order	Error	C-order
8	7.5634e-03	–	7.6367e-03	–	8.0242e-03	–	7.8281e-03	–
16	5.5244e-04	3.7751	5.7566e-04	3.7297	6.0536e-04	3.7285	5.7222e-04	3.7740
32	3.6423e-05	3.9229	4.0845e-05	3.8170	4.1833e-05	3.8551	3.8380e-05	3.8981
64	2.5410e-06	3.8414	2.6233e-06	3.9607	2.6845e-06	3.9619	2.4484e-06	3.9705
128	1.7600e-07	3.8517	1.6294e-07	4.0090	1.6821e-07	3.9963	1.5374e-07	3.9932

Table: The average (error) obtained for Problem 2 with $h = \Delta t$, $K_1 = 0.6$ and $K_2 = 0.4$.

$n_x + 1$	$\alpha = 1.2$		$\alpha = 1.4$		$\alpha = 1.6$		$\alpha = 1.8$	
	Error	C-order	Error	C-order	Error	C-order	Error	C-order
8	5.2818e-03	–	5.9446e-03	–	6.9026e-03	–	7.5938e-03	–
16	3.6813e-04	3.8427	4.4073e-04	3.7536	5.3561e-04	3.6878	5.7954e-04	3.7118
32	2.4648e-05	3.9007	3.1593e-05	3.8022	3.7389e-05	3.8405	3.9293e-05	3.8826
64	1.9070e-06	3.6921	2.0207e-06	3.9667	2.3879e-06	3.9688	2.5031e-06	3.9725
128	1.3462e-07	3.8243	1.2490e-07	4.0160	1.4801e-07	4.0120	1.5598e-07	4.0043

Numerical results

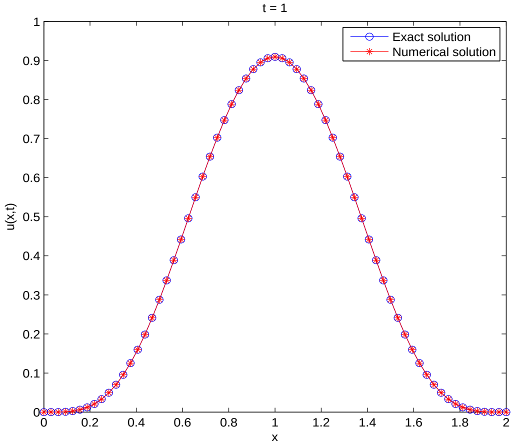

Figure: The behavior solution from the proposed methods at $t = 1$ with $h = \Delta t = 1/64$, $\alpha = 1.5$, $K_1 = 0.3$ and $K_2 = 0.7$.

Numerical results

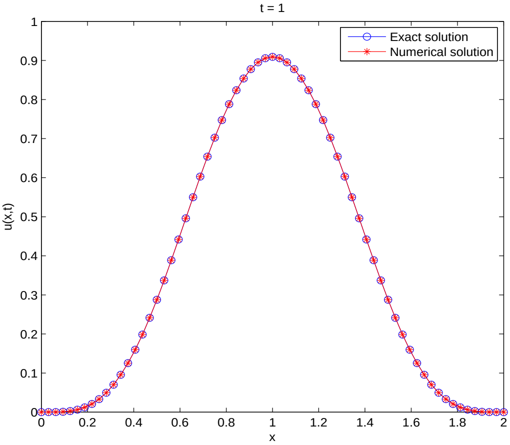

Figure: The behavior solution from the proposed methods at $t = 1$ with $h = \Delta t = 1/64$, $\alpha = 1.8, K_1 = 0.6$ and $K_2 = 0.4$.

Numerical results

Example 3. Consider one-side fractional diffusion equations (3)

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = K_{1\,a}D_x^{\alpha} u(x,t) + f(x,t), & (x,t) \in (0,2) \times (0,1], \\ u(x,0) = x^{4+|\alpha-1.5|/2}, & x \in [0,2], \\ u(0,t) = 0, \quad u(2,t) = \phi(t), & t \in [0,1], \end{cases}$$

with the diffusion coefficient $K_1 = 1$, $\phi(t) = e^{-t}(2^{4+|\alpha-1.5|/2})$ and the source term

$$f(x,t) = -e^{-t}\left(x^{4+|\alpha-1.5|/2} + \frac{\Gamma(5+|\alpha-1.5|/2)}{\Gamma(5-\alpha+|\alpha-1.5|/2)} x^{4-\alpha+|\alpha-1.5|/2}\right).$$

The exact solution is $u(x,t) = e^{-t}x^{4+|\alpha-1.5|/2}$.

Numerical results

Table: The average (error) obtained for Problem 3 with $h = \Delta t$ and $t = 1$.

$n_x + 1$	$\alpha = 1.2$ Error	C-order	$\alpha = 1.4$ Error	C-order	$\alpha = 1.6$ Error	C-order	$\alpha = 1.8$ Error	C-order
8	3.5363e-05	–	2.3994e-04	–	2.5929e-04	–	1.8967e-04	–
16	2.2357e-06	3.9835	1.7066e-05	3.8135	1.7475e-05	3.8912	1.3251e-05	3.8393
32	1.4568e-07	3.9398	1.1364e-06	3.9086	1.1168e-06	3.9678	8.7745e-07	3.9167
64	1.0072e-08	3.8544	7.3160e-08	3.9573	6.9642e-08	4.0033	5.7097e-08	3.9418
128	6.9664e-10	3.8538	4.6194e-09	3.9853	4.2878e-09	4.0216	3.6738e-09	3.9581

Table: The average (error) obtained for Problem 3 with $h = \Delta t$ and $t = 1$.

$n_x + 1$	$\alpha = 1.3$ Error	C-order	$\alpha = 1.5$ Error	C-order	$\alpha = 1.7$ Error	C-order	$\alpha = 1.9$ Error	C-order
8	1.5050e-04	–	2.7910e-04	–	2.2771e-04	–	1.4845e-04	–
16	1.1206e-05	3.7475	1.9551e-05	3.8355	1.4770e-05	3.9464	1.0307e-05	3.8483
32	7.6408e-07	3.8744	1.2985e-06	3.9123	9.7555e-07	3.9203	6.9435e-07	3.8918
64	4.9762e-08	3.9406	8.4091e-08	3.9488	6.2883e-08	3.9555	4.5737e-08	3.9242
128	3.1542e-09	3.9797	5.3737e-09	3.9680	4.0269e-09	3.9649	2.9753e-09	3.9423

Numerical results

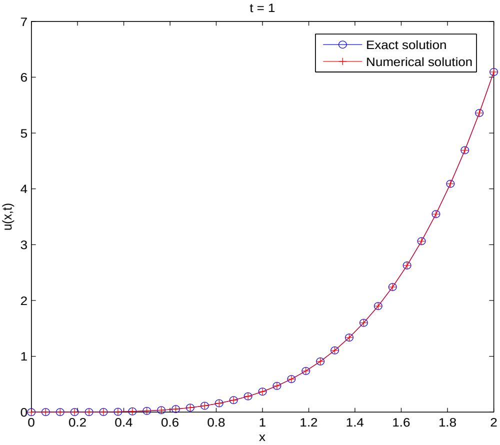

Figure: The behavior solution from the proposed methods at $t = 1$ with $h = \Delta t = 1/32$, $\alpha = 1.4$.

Numerical results

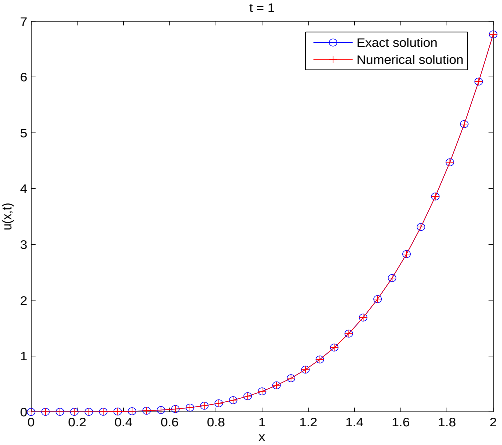

Figure: The behavior solution from the proposed methods at $t = 1$ with $h = \Delta t = 1/32$, $\alpha = 1.9$.

Summary

Conclusions:

- The WSGD to discretize the Riemann-Liouville fractional differential operator, which is fourth order accurate in space;
- Unconditionally stable BVMs are used, which is also fourth order accurate in time;
- To solve such linear system, some efficient iterative solution techniques have been obtained;
- The proposed high order numerical methods can also be extended to efficiently solve the space-fractional telegraph equation and the space tempered fractional diffusion equation.

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Q & A

Thank you for attention!